Certain Two-Dimensional Local Limit Theorems with Applications to the Statistics of Convex Polygons

Sakhavet M. Zarbaliev

Department of Mathematics, Econometrics and Information Technology,
Moscow State Institute of International Relations (MGIMO University),
pr. Vernadskogo 76, Moscow, 119454 Russia.
E-mail: s.m.zarbaliev@inno.mgimo.ru

Abstract

The work is devoted to asymptotic properties of the ensemble of integer convex broken lines with respect to some classes of probability distributions generated by multiplicative measures.

Keywords: Convex lattice polygonal line; Limit shape; Multiplicative measures; Local limit theorem.

Consider the collection Π of *convex lattice polygons* Γ on

$$\mathbb{Z}_+^2 := \{(k_1, k_2) \in \mathbb{Z}^2 \colon k_j \geq 0\},$$

starting at $(0, 0)$ and such that the inclination of their consecutive edges grows staying no longer than $\pi/2$ (clearly, any such Γ lies within the first coordinate quadrant). Let Π_{m_1,m_2} denote the subset of obtained by fixing the right endpoint of polygons at (m_1, m_2). The asymptotical properties of Π_{m_1,m_2} as $m_1, m_2 \to \infty,\ m_2/m_1 \to c \in (0, \infty)$, in particular the asymptotics of the number of polygons in Π_{m_1,m_2} and their limiting shape, have been studied via quite different methods in Vershik [6], and Barany [1].

A probabilistic method proposed in the seminal paper by Sinai [5] proceeds from the idea to introduce on Π an appropriate probability distribution Q_z depending on the parameters $z = (z_1, z_2)$ and then induce the corresponding conditional distribution P_{m_1,m_2} on Π_{m_1,m_2}. It turns out that one can choose the parameters z_1, z_2 in such a way that distribution P_{m_1,m_2} proves to be uniform on Π_{m_1,m_2}, which allows one to relate the probabilistic and combinatorial properties of Π_{m_1,m_2}. Along this line, one proves the required limit theorems (e.g., the law of large numbers) for the ensemble Π with respect to the distribution Q_z, which are then carried over to Π_{m_1,m_2} and P_{m_1,m_2}. At the latter step, the key role is played by a local limit theorem for the probability $Q_z\{\Gamma \in \Pi_{m_1,m_2}\}$. Note that, in terms of Statistical

Mechanics, this approach amounts to the description of $\mathit{\Pi}$ and $\mathit{\Pi}_{m_1,m_2}$ as a grand canonical and microcanonical ensembles equipped with the Gibbs measures Q_z and P_{m_1,m_2}, respectively.

Let us describe our setting more specifically. Consider the set

$$\mathcal{X} := \{x = (x_1, x_2) \in \mathbb{Z}_+^2 : \ \gcd(x_1, x_2) = 1\} \tag{1}$$

of all pairs $x = (x_1, x_2)$ of coprime non-negative integers and let $\Phi_0(\mathcal{X})$ be the space of finite functions $\nu : \mathcal{X} \to \mathbb{Z}_+$. As explained in [5], there is a bijection between $\Phi_0(\mathcal{X})$ and $\mathit{\Pi}$. Namely, given $\nu \in \Phi_0(\mathcal{X})$, arrange the pairs $(x_1, x_2) \in \operatorname{supp} \nu$ according to the increase of the slope x_2/x_1. Multiplying the vectors (x_1, x_2) by the corresponding values $\nu(x) > 0$ yields the consecutive edges of a convex polygon. The converse procedure is similar.

Define the probability measure on $\Phi_0(\mathcal{X})$ by

$$Q_z\{\nu\} := \prod_{x \in \mathcal{X}} (z_1^{x_1} z_2^{x_2})^{\nu(x)} (1 - z_1^{x_1} z_2^{x_2}), \tag{2}$$

where $z_i := z_i(x)$, $x \in \mathcal{X}$, and $z = (z_1, z_2)$. As usual, Q_z can be extended to the space $\Phi_0(\mathcal{X})$ of all non-negative functions on $\mathcal{X}$. However, in fact $Q_z\{\nu \in \Phi_0(\mathcal{X})\}$. Note also that $\{\nu(x),\, x \in \mathcal{X}\}$ are independent geometrical random variables (with parameters $z_1^{x_1} z_2^{x_2}$, respectively, and thus in general non-identically distributed). The coordinates of the polygons right endpoint are given by $\xi_i := \sum_{x \in \mathcal{X}} x_i \nu(x)\, (i = 1, 2)$. Thus, the corresponding conditional distribution on $\mathit{\Pi}_{m_1,m_2}$ can be written in the form

$$P_{m_1,m_2}\{\mathit{\Gamma}\} := \frac{Q_z\{\mathit{\Gamma},\, \mathit{\Gamma} \in \mathit{\Pi}_{m_1,m_2}\}}{Q_z\{\xi_1 = m_1,\, \xi_2 = m_2\}}. \tag{3}$$

From (3) it is seen that one needs to know the asymptotic behavior of the denomination $Q_z\{\xi_1 = m_1,\, \xi_2 = m_2\}$ as $m_1, m_2 \to \infty$.

Let E_z and D_z stand respectively for the expectation and variance with respect to the law Q_z . For $i = 1, 2$ denote $A_i = E_z \xi_i$, $B_i^2 := D_z \xi_i$, and introduce the Lyampunovs ratios

$$L_i := B_i^{-3} \sum_{x \in \mathcal{X}} x_i^3\, E_z |\nu(x) - E_z \nu(x)|^3.$$

Theorem 1 *Uniformly in $k_1, k_2 \geq 0$ the inequality holds*

$$\begin{aligned} &|B_1 B_2 \cdot Q_z\{\xi_1 = k_1, \xi_2 = k_2\} - \varphi(y_1, y_2)| \\ &\qquad \leq c_1 L_1 + c_2 L_2 + c_{12} L_1 L_2 + c_3 m_1^{1/3} \exp\{-c_4 m_1^{1/3}\}. \end{aligned}$$

Here $c_1, \ldots, c_4$ *are absolute constants,* $y_i = (k_i - A_i)/B_i$ $(i = 1, 2)$*, and* $\varphi(y_1, y_2)$ *is the destiny of normal distribution with the zero mean and covariance matrix*

$$\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

where ρ *is the coefficient of correlation between* ξ_1, ξ_2*.*

Theorem 2 *Suppose that the equalitions* $E_z \xi_i = m_i$ $(i = 1, 2)$ *hold and* $\rho \to \rho_0 < 1$ *as*

$$m_1, m_2 \to \infty, \; m_2/m_1 \to c \in (0, \infty).$$

Set $\varphi_0 = \varphi|_{\rho=\rho_0}$*. If* $L_1, L_2 \to 0$*, then uniformly in* $k_1, k_2 \geq 0$

$$|B_1 B_2 \cdot Q_z\{\xi_1 = k_1, \xi_2 = k_2\} - \varphi_0(y_1, y_2)| \to 0.$$

Now let the z_i's be of the form

$$z_i(x) = \exp\{-m_1^{-1/3} \delta_i(x_2/x_1)\}$$

with some functions $\delta_i > 0\,(i = 1, 2)$. As shown in [5], the conditions

$$E_z \xi_1 = m_1, \; E_z \xi_2 = m_2$$

can be satisfied with $\delta_1, \delta_2 \equiv$ *const*. Moreover, in this case one can check that $\rho \to 1/2$ and $L_1, L_2 \to 0$, so that Theorem 2 applies. In particular,

$$Q_z\{\xi_1 = m_1, \xi_2 = m_2\} \sim (2\sqrt{3}\pi)^{-1} m_1^{-4/3}.$$

We apply the above approach to the problem of approximation of convex functions $f \in C^2$ via convex polygons [2], [3] . In that case, the functions δ_1, δ_2 are chosen from the equation

$$E_z[\sum_{x_2/x_1 \leq s} x_2 \nu(x)] = m_1 f(u(s)), \quad s \in [0, \infty).$$

Remark 1 *Our results can be extended to the case where* Q_z *is taken in the form*

$$Q_z\{\nu\} = \prod_{x \in \mathcal{X}} h(\nu(x)) (z_1^{x_1} z_2^{x_2})^{\nu(x)} \, G(x, z)^{-1}$$

(cf. (1)), where $h > 0$ *is a function such that*

$$G(x, z) := \sum_{k=1}^{\infty} h(k) (z_1^{x_1} z_2^{x_2})^k < \infty.$$

Remark 2 *We conclude with a conjecture that multidimensional analogues of Theorems 1, 2 are valid, although the corresponding geometrical interpretation seems to be difficult to provide.*

Acknowledgments

The author is greatly indebted to his scientific adviser Prof. Ya. G. Sinai for posing the problem for simulating interest in the work. He is also thankful to L. V. Bogachev for valuable discussions and remarks. The author would also like to thank Faculty of Department of Mathematics of University of Houston-Clear Lake for invitation and their hospitality.

References

[1] I. Bárány, The limit shape of convex lattice polygons, Discrete Comput. Geom. 13 (1995) 279–295; 1318778.

[2] L.V. Bogachev, S.M. Zarbaliev, On the approximation of convex functions by random polygonal lines, Dokl. Akad. Nauk 364 (3) (1999) 299–302 (in Russian); English transl.: Approximation of convex functions by random polygonal lines, Doklady Math. 59 (1999) 46–49; 1706217.

[3] L.V. Bogachev, S.M. Zarbaliev, A proof of the Vershik–Prokhorov conjecture on the universality of the limit shape for a class of random polygonal lines, Dokl. Akad. Nauk 425 (3) (2009) 299–304 (in Russian); English transl. in Doklady Math. 79 (2009) 197–202; 2541116.

[4] L.V. Bogachev, S.M. Zarbaliev, Inverse problem of the limit shape for convex lattice polygonal lines, Preprint (2011), http://arxiv.org/abs/1110.6636 (last accessed 01.11.2011).

[5] Ya.G. Sinai, A probabilistic approach to the analysis of the statistics of convex polygonal lines, Funktsional. Anal. i Prilozhen. 28 (2) (1994) 41–48 (in Russian); English transl.: Probabilistic approach to the analysis of statistics for convex polygonal lines, Funct. Anal. Appl. 28 (1994) 108–113; 1283251.

[6] A.M. Vershik, The limit form of convex integral polygons and related problems, Funktsional. Anal. i Prilozhen. 28 (1) (1994) 16–25 (in Russian); English transl.: The limit shape of convex lattice polygons and related topics, Funct. Anal. Appl. 28 (1994) 13–20; 1275724.